Geom-SPIDER-EM: Faster Variance Reduced Stochastic Expectation Maximization for Nonconvex Finite-Sum Optimization

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GEOM-SPIDER-EM: FASTER VARIANCE REDUCED STOCHASTIC EXPECTATION MAXIMIZATION FOR NONCONVEX FINITE-SUM OPTIMIZATION

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ABSTRACT

The Expectation Maximization (EM) algorithm is a key reference for inference in latent variable models; unfortunately, its computational cost is prohibitive in the large scale learning setting. In this paper, we propose an extension of the Stochastic Path-Integrated Differential EstimatoR EM (SPIDER-EM) and derive complexity bounds for this novel algorithm, designed to solve smooth nonconvex finite-sum optimization problems. We show that it reaches the same state of the art complexity bounds as SPIDER-EM; and provide conditions for a linear rate of convergence. Numerical results support our findings.

***Index Terms*—** Large scale learning, Latent variable analysis, Expectation Maximization, Nonconvex stochastic optimization, Variance reduction.

1. INTRODUCTION

Intelligent processing of large data sets and efficient learning of high-dimensional models require new optimization algorithms designed to be robust to big data and complex models era (see e.g. [1–3]). This paper is concerned with stochastic optimization of a nonconvex finite-sum smooth objective function

$$\mathrm{Argmin}_{\theta\in\Theta} F(\theta), \qquad F(\theta) \stackrel{\mathrm{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\theta) + \mathsf{R}(\theta) \;, \quad (1)$$

when $\Theta \subseteq \mathbb{R}^d$ and F cannot be explicitly evaluated (nor its gradient). Many statistical learning problems can be cast into this framework, where n is the number of observations or examples, $\mathcal{L}_i$ is a loss function associated to example $\#i$ (most often, a negative log-likelihood), and R is a penalty term promoting sparsity, regularity, etc. Empirical risk minimization in machine learning is a matter for (1). Intractability of $F(\theta)$ might come from two sources. The first, referred to as *large scale learning* setting, is that the number n is very large so that the computations involving a sum over n terms should be either simply avoided or sparingly used during the run of the optimization algorithm (see e.g. [4] for an introduction to the bridge between large scale learning and stochastic approximation; see [5,6] for applications to training of deep neural networks for signal and image processing). The second is due to the presence of latent variables: for any i, the function $\mathcal{L}_i$ is a (high-dimensional) integral over latent variables. Such a latent variable context is a classical statistical modeling: for example as a tool for solving inference in mixture models [7], for the definition of mixed models capturing variability among examples [8] or for modeling hidden and/or missing variables (see e.g. applications in text modeling through latent Dirichlet allocation [9], in audio source separation [10, 11], in hyper-spectral imaging [12]).

In this contribution, we address the two levels of intractability in the case $\mathcal{L}_i$ is of the form

$$\mathcal{L}_i(\theta) \stackrel{\mathrm{def}}{=} -\log \int_{\mathsf{Z}} h_i(z) \exp\left(\langle s_i(z), \phi(\theta)\rangle\right) \mu(\mathrm{d}z) \;. \quad (2)$$

This setting in particular covers the case when $\sum_{i=1}^n \mathcal{L}_i(\theta)$ is the negated log-likelihood of the observations $(Y_1, \cdots, Y_n)$, the pairs observation/latent variable $\{(Y_i, Z_i), i \le n\}$ are independent, and the distribution of the latent variable given the observation Y_i, given by $z \mapsto h_i(z) \exp\left(\langle s_i(z), \phi(\theta)\rangle\right) \mu(\mathrm{d}z)$ up to a multiplicative constant, is from the curved exponential family. Gaussian mixture models are typical examples, as well as mixtures of distributions from the curved exponential family. In the framework (1)-(2), a Majorize-Minimization approach through the Expectation-Maximization (EM) algorithm [13] is standard; unfortunately, the computational cost of the batch EM can be prohibitive in the large scale learning setting. Different strategies were proposed to address this issue [14–18]: they combine mini-batches processing, Stochastic Approximation (SA) techniques (see e.g. [19, 20]) and variance reduction methods.

The first contribution of this paper is to provide a novel algorithm, the generalized Stochastic Path-Integrated Differential EstimatoR EM (`g-SPIDER-EM`), which is among the variance reduced stochastic EM methods for nonconvex finite-sum optimization of the form (1)-(2); the generalizations allow a reduced computational cost without altering the convergence properties. The second contribution is the proof of complexity bounds, that is the number of parameter updates (M-step) and the number of conditional expectations evaluations (E-step), in order to reach ϵ-approximate stationary points; these bounds are derived for a specific form of `g-SPIDER-EM`: we show that the complexity bounds are the same as those of `SPIDER-EM`, bounds which are state of the art ones and overpass all the previous ones. Linear convergence rates are proved under a Polyak-Łojasiewicz condition. Finally, numerical results support our findings and provide insights on how to implement `g-SPIDER-EM` in order to inherit the properties of `SPIDER-EM` while reducing the computational cost.

Notations For $a, b \in \mathbb{R}^q$, $\langle a, b\rangle$ is the scalar product, and $\|\cdot\|$ the associated norm. For a matrix A, A^T is its transpose. For a positive integer n, set $[n]^\star \stackrel{\mathrm{def}}{=} \{1, \cdots, n\}$ and $[n] \stackrel{\mathrm{def}}{=} \{0, \cdots, n\}$. ∇f denotes the gradient of a differentiable function f. The minimum of a and b is denoted by $a \wedge b$. Finally, we use standard big O notation

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to leave out constants. For a random variable U and/or a filtration $\mathcal{F}$, $\sigma(U, \mathcal{F})$ denotes the sigma algebra generated by U and $\mathcal{F}$.

2. EM-BASED METHODS IN THE EXPECTATION SPACE

We begin by formulating the model assumptions:

A1. *$\Theta \subseteq \mathbb{R}^d$ is a convex set. $(\mathsf{Z}, \mathcal{Z})$ is a measurable space and μ is a σ-finite positive measure on $\mathcal{Z}$. The functions $\mathsf{R} : \Theta \to \mathbb{R}$, $\phi : \Theta \to \mathbb{R}^q$, $s_i : \mathsf{Z} \to \mathbb{R}^q$, $h_i : \mathsf{Z} \to \mathbb{R}_+$ for all $i \in [n]^\star$ are measurable. For any $\theta \in \Theta$ and $i \in [n]^\star$, $|\mathcal{L}_i(\theta)| < \infty$.*

For any $\theta \in \Theta$ and $i \in [n]^\star$, define the posterior density of the latent variable Z_i given the observation Y_i:

$$z \mapsto p_i(z;\theta) \stackrel{\text{def}}{=} h_i(z) \exp\left(\langle s_i(z), \phi(\theta)\rangle + \mathcal{L}_i(\theta)\right) , \quad (3)$$

note that the dependence upon y_i follows through the index i in the above. Set

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} \int_{\mathsf{Z}} s_i(z)\, p_i(z;\theta)\mu(\mathrm{d}z), \quad \bar{s}(\theta) \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^n \bar{s}_i(\theta) . \quad (4)$$

A2. *The expectations $\bar{s}_i(\theta)$ are well defined for all $\theta \in \Theta$ and $i \in [n]^\star$. For any $s \in \mathbb{R}^q$, $\mathrm{Argmin}_{\theta\in\Theta}\ (-\langle s, \phi(\theta)\rangle + \mathsf{R}(\theta))$ is a (non empty) singleton denoted by $\{\mathsf{T}(s)\}$.*

EM is an iterative algorithm: given a current value $\tau_k \in \Theta$, the next value is $\tau_{k+1} \leftarrow \mathsf{T} \circ \bar{s}(\tau_k)$. It combines an expectation step which boils down to the computation of $\bar{s}(\tau_k)$, an expectation under $p(\cdot;\tau_k)$; and a maximization step through the computation of the map T. Equivalently, by using T which maps $\mathbb{R}^q$ to Θ, it can be described in the *expectation space* (see [21]): given the current value $\bar{s}^k \in \bar{s}(\Theta)$, the next value is $\bar{s}^{k+1} \leftarrow \bar{s} \circ \mathsf{T}(\bar{s}^k)$. In this paper, we see EM as an iterative algorithm operating in the expectation space. In that case, the fixed points of the EM operator $\bar{s} \circ \mathsf{T}$ are the roots of the function h

$$h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s . \quad (5)$$

EM possesses a Lyapunov function: in the parameter space, it is the objective function F where by definition of the EM sequence, it holds $F(\tau_{k+1}) \le F(\tau_k)$; in the expectation space, it is $\mathrm{W} \stackrel{\text{def}}{=} F \circ \mathsf{T}$, and $\mathrm{W}(\bar{s}^{k+1}) \le \mathrm{W}(\bar{s}^k)$ holds. In order to derive complexity bounds, regularity assumptions are required on W:

A 3. *The functions ϕ and R are continuously differentiable on Θ^v, where Θ^v is an open neighborhood of Θ when Θ is not open and $\Theta^v \stackrel{\text{def}}{=} \Theta$ otherwise. T is continuously differentiable on $\mathbb{R}^q$. The function F is continuously differentiable on Θ^v and for any $\theta \in \Theta$, $\nabla F(\theta) = -\nabla\phi(\theta)^T \bar{s}(\theta) + \nabla\mathsf{R}(\theta)$. For any $s \in \mathbb{R}^q$, $B(s) \stackrel{\text{def}}{=} \nabla(\phi \circ \mathsf{T})(s)$ is a symmetric $q \times q$ matrix and there exist $0 < v_{min} \le v_{max} < \infty$ such that for all $s \in \mathbb{R}^q$, the spectrum of $B(s)$ is in $[v_{min}, v_{max}]$. For any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i. The function $s \mapsto \nabla(F \circ \mathsf{T})(s) = -B(s)\,(\bar{s} \circ T(s) - s)$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_{W}.*

A3 implies that W has globally Lipschitz gradient and $\nabla W(s) = -B(s)h(s)$ for some positive definite matrix $B(s)$ (see e.g. [21, Lemma 2]; see also [22, Propositions 1 and 2]). Note that this implies that $\nabla W(s^\star) = 0$ iff $h(s^\star) = 0$.

Unfortunately, in the large scale learning setting (when $n \gg 1$), EM can not be easily applied since each iteration involves n conditional expectations evaluations through $\bar{s} = n^{-1}\sum_{i=1}^n \bar{s}_i$. Incremental EM techniques have been proposed to address this issue: the most straightforward approach amounts to use a SA scheme with mean field h since. Upon noting that $h(s) = \mathbb{E}\left[\bar{s}_I \circ \mathsf{T}(s)\right] - s$ where I is a uniform random variable (r.v.) on $[n]^\star$, the fixed points of the EM operator $\bar{s} \circ \mathsf{T}$ are those of the SA scheme

$$\widehat{S}_{k+1} = \widehat{S}_k + \gamma_{k+1}\left(\mathsf{b}^{-1}\sum_{i\in\mathcal{B}_{k+1}} \bar{s}_i \circ \mathsf{T}(\widehat{S}_k) - \widehat{S}_k\right) \quad (6)$$

where $\{\gamma_k, k \ge 0\}$ is a deterministic positive step size sequence, and $\mathcal{B}_{k+1}$ is sampled from $[n]^\star$ independently from the past of the algorithm. This forms the basis of `Online-EM` proposed by [15] (see also [23]). Variance reduced versions were also proposed and studied: Incremental EM (`i-EM`) [14, 24], Stochastic EM with variance reduction (`sEM-vr`) [16], Fast Incremental EM [17, 22] (`FIEM`) and more recently, Stochastic Path-Integrated Differential EstimatoR EM (`SPIDER-EM`) [18].

As shown in [22, section 2.3], these algorithms can be seen as a combination of SA with *control variate*: upon noting that $h(s) = h(s) + \mathbb{E}[U]$ for any r.v. U such that $\mathbb{E}[U] = 0$, *control variates within SA* procedures replace (6) with

$$\widehat{S}_{k+1} = \widehat{S}_k + \gamma_{k+1}\left(\mathsf{b}^{-1}\sum_{i\in\mathcal{B}_{k+1}} \bar{s}_i \circ \mathsf{T}(\widehat{S}_k) + U_{k+1} - \widehat{S}_k\right)$$

for a choice of U_{k+1} such that the new algorithm has better properties (for example, in terms of complexity - see the end of Section 3).

Lastly, we remark that A1–A3 are common assumptions (see e.g. [18] and [22] and references therein).

3. THE GEOM-SPIDER-EM ALGORITHM

Data: $k_{\text{out}} \in \mathbb{N}^\star$; $\widehat{S}_{\text{init}} \in \mathbb{R}^q$; $\xi_t \in \mathbb{N}^\star$ for $t \in [k_{\text{out}}]^\star$; $\gamma_{t,0} \ge 0$, $\gamma_{t,k} > 0$ for $t \in [k_{\text{out}}]^\star$, $k \in [\xi_t]^\star$.
Result: The g-SPIDER-EM sequence: $\{\widehat{S}_{t,k}\}$

$\widehat{S}_{1,0} = \widehat{S}_{1,-1} = \widehat{S}_{\text{init}}$;
$\mathsf{S}_{1,0} = \bar{s} \circ \mathsf{T}(\widehat{S}_{1,-1}) + \mathcal{E}_1$;
for $t = 1, \cdots, k_{\text{out}}$ **do**
  for $k = 0, \ldots, \xi_t - 1$ **do**
    Sample a mini batch $\mathcal{B}_{t,k+1}$ of size b from $[n]^\star$;
    $\mathsf{S}_{t,k+1} = \mathsf{S}_{t,k} + \mathsf{b}^{-1}\sum_{i\in\mathcal{B}_{t,k+1}}\left(\bar{s}_i \circ \mathsf{T}(\widehat{S}_{t,k}) - \bar{s}_i \circ \mathsf{T}(\widehat{S}_{t,k-1})\right)$;
    $\widehat{S}_{t,k+1} = \widehat{S}_{t,k} + \gamma_{t,k+1}\left(\mathsf{S}_{t,k+1} - \widehat{S}_{t,k}\right)$
  $\widehat{S}_{t+1,-1} = \widehat{S}_{t,\xi_t}$;
  $\mathsf{S}_{t+1,0} = \bar{s} \circ \mathsf{T}(\widehat{S}_{t+1,-1}) + \mathcal{E}_{t+1}$;
  $\widehat{S}_{t+1,0} = \widehat{S}_{t+1,-1} + \gamma_{t+1,0}\left(\mathsf{S}_{t+1,0} - \widehat{S}_{t+1,-1}\right)$

Algorithm 1: The `g-SPIDER-EM` algorithm. The $\mathcal{E}_t$'s are introduced as a perturbation to the computation of $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,-1})$; they can be null.

The algorithm *generalized Stochastic Path-Integrated Differential EstimatoR Expectation Maximization* (`g-SPIDER-EM`) described by Algorithm 1 uses a new strategy when defining the approximation of $\bar{s} \circ \mathsf{T}(s)$ at each iteration. It is composed of nested loops: k_{out} outer loops, each of them formed with a possibly random number of inner loops. Within the tth outer loop, `g-SPIDER-EM` mimics the identity $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) = \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1}) + \{\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1})\}$. More precisely, at iteration $k+1$, the approximation

$\mathsf{S}_{t,k+1}$ of the full sum $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k})$ is the sum of the current approximation $\mathsf{S}_{t,k}$ and of a Monte Carlo approximation of the difference (see Lines 5, 6, in Algorithm 1); the examples i in $\mathcal{B}_{t,k+1}$ used in the approximation of $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k})$ and those used for the approximation of $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1})$ are the same - which make the approximations correlated and favor a variance reduction when plugged in the SA update (Line 7). $\mathcal{B}_{t,k+1}$ is sampled with or without replacement; even when $\mathcal{B}_{t,k+1}$ collects independent examples sampled uniformly from $[n]^\star$, we have $\mathbb{E}\left[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}\right] - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) = \mathsf{S}_{t,k} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1})$ where $\mathcal{F}_{t,k}$ is the sigma-field collecting the randomness up to the end of the outer loop $\#t$ and inner loop $\#k$: the approximation $\mathsf{S}_{t,k+1}$ of $\bar{s} \circ \mathsf{T}(\widehat{S}_{t,k})$ is biased - a property which makes the theoretical analysis of the algorithm challenging. This approximation is reset (see Lines 2,9) at the end of an outer loop: in the "standard" `SPIDER-EM`, $\mathsf{S}_{t,0} = \bar{s} \circ \mathsf{T}(\widehat{S}_{t,-1})$ is computed, but this "refresh" can be only partial, by computing an update on a (large) batch $\tilde{\mathcal{B}}_{t,0}$ (size $\tilde{\mathsf{b}}_t$) of observations: $\mathsf{S}_{t,0} = \tilde{\mathsf{b}}_t^{-1} \sum_{i \in \tilde{\mathcal{B}}_{t,0}} \bar{s}_i \circ \mathsf{T}(\widehat{S}_{t,-1})$. Such a reset starts a so-called *epoch* (see Line 3). The number of inner loops ξ_t at epoch $\#t$ can be deterministic; or random, such as a uniform distribution on $[k_{\rm in}]^\star$ or a geometric distribution, and drawn prior the run of the algorithm.

Comparing `g-SPIDER-EM` with `SPIDER-EM` [18], we notice that *(i)* the former allows a perturbation $\mathcal{E}_t$ when initializing $\mathsf{S}_{t,0}$, which is important for computational cost reduction; *(ii)* `g-SPIDER-EM` considers epochs with time-varying length ξ_t which covers situations when it is random and chosen independently of the other sources of randomness (the errors $\mathcal{E}_t$, the mini batches $\mathcal{B}_{t,k+1}$). Hereafter, we provide an original analysis of an `g-SPIDER-EM`, namely `Geom-SPIDER-EM` which corresponds to the case $\xi_t \leftarrow \Xi_t, \Xi_t$ being a geometric r.v. on $\mathbb{N}^\star$ with success probability $1 - \rho_t \in (0,1)$: $\mathbb{P}(\Xi_t = k) = (1-\rho_t)\rho_t^{k-1}$ for $k \geq 1$ (hereafter, we will write $\Xi_t \sim \mathcal{G}^\star(1-\rho_t)$). Since Ξ_t is also the first success distribution in a sequence of independent Bernoulli trials, the geometric length could be replaced with: *(i)* at each iteration k of epoch t, sample a Bernoulli r.v. with a probability of success $(1-\rho_t)$; *(ii)* when the coin comes up head, start a new epoch (see [25,26] for similar ideas on stochastic gradient algorithms).

Let us establish complexity bounds for `Geom-SPIDER-EM`. We analyze a randomized terminating iteration $\Xi^\star$ [27] and discuss how to choose $k_{\rm out}, \mathsf{b}$ and $\xi_1, \cdots, \xi_{k_{\rm out}}$ as a function of the batch size n and an accuracy $\epsilon > 0$ to reach ϵ-approximate stationarity i.e. $\mathbb{E}[\|h(\widehat{S}_{\Xi^\star})\|^2] \leq \epsilon$. To this end, we endow the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the sigma-fields $\mathcal{F}_{1,0} \overset{\rm def}{=} \sigma(\mathcal{E}_1)$, $\mathcal{F}_{t,0} \overset{\rm def}{=} \sigma(\mathcal{F}_{t-1,\xi_t}, \mathcal{E}_t)$ for $t \geq 2$, and $\mathcal{F}_{t,k+1} \overset{\rm def}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$ for $t \in [k_{\rm out}]^\star, k \in [\xi_t - 1]$. For a r.v. $\Xi_t \sim \mathcal{G}^\star(1-\rho_t)$, set $\overline{\mathbb{E}}_t[\phi(\Xi_t)|\mathcal{F}_{t,0}] \overset{\rm def}{=} (1-\rho_t)\sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k)|\mathcal{F}_{t,0}]$ for any bounded measurable function ϕ.

Theorem 1. *Assume A1 to A3. For any $t \in [k_{\rm out}]^\star$, let $\rho_t \in (0,1)$ and $\Xi_t \sim \mathcal{G}^\star(1-\rho_t)$. Run Algorithm 1 with $\gamma_{t,k+1} = \gamma_t > 0$, $\gamma_{1,0} = 0$ and $\xi_t \leftarrow \Xi_t$ for any $t \in [k_{\rm out}]^\star$, $k \geq 0$. Then, for any $t \in [k_{\rm out}]^\star$,*

$$\begin{aligned}
&\frac{v_{\min}\gamma_t}{2(1-\rho_t)} \overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0}\right] \\
&\leq \mathrm{W}(\widehat{S}_{t,0}) - \overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right] + \frac{v_{\max}\gamma_t}{2(1-\rho_t)}\|\mathcal{E}_t\|^2 \\
&+ \frac{v_{\max}\gamma_t\gamma_{t,0}^2}{2(1-\rho_t)}\frac{L^2}{\mathsf{b}}\|\Delta\widehat{S}_{t,0}\|^2 + \mathcal{N}_t\,\overline{\mathbb{E}}_t\left[\|\Delta\widehat{S}_{t,\Xi}\|^2|\mathcal{F}_{t,0}\right] \;;
\end{aligned}$$

where $\Delta\widehat{S}_{t,\xi} \overset{\rm def}{=} \mathsf{S}_{t,\xi} - \widehat{S}_{t,\xi-1}$, $L^2 \overset{\rm def}{=} n^{-1}\sum_{i=1}^n L_i$, and

$$\mathcal{N}_t \overset{\rm def}{=} -\frac{\gamma_t}{2(1-\rho_t)}\left(v_{\min} - \gamma_t L_{\dot{\mathrm{W}}} - \frac{v_{\max}L^2\rho_t}{(1-\rho_t)\mathsf{b}}\gamma_t^2\right) .$$

Theorem 1 is the key result from which our conclusions are drawn; its proof is adapted from [18, section 8] (also see [28, Theorem 10]).

Let us discuss the rate of convergence and the complexity of `Geom-SPIDER-EM` in the case: for any $t \in [k_{\rm out}]^\star$, the mean number of inner loops is $(1-\rho_t)^{-1} = k_{\rm in}$, $\gamma_{t,0} = 0$ and $\gamma_t = \alpha/L$ for $\alpha > 0$ satisfying

$$v_{\min} - \alpha\frac{L_{\dot{\mathrm{W}}}}{L} - \alpha^2 v_{\max}\frac{k_{\rm in}}{\mathsf{b}}\left(1 - \frac{1}{k_{\rm in}}\right) > 0 .$$

Linear rate. When $\Xi \sim \mathcal{G}^\star(1-\rho)$, we have (see [28, Lemma 1])

$$\rho\mathbb{E}\left[D_\Xi\right] \leq \rho\mathbb{E}\left[D_\Xi\right] + (1-\rho)D_0 = \mathbb{E}\left[D_{\Xi-1}\right] \tag{7}$$

for any positive sequence $\{D_k, k \geq 0\}$; Theorem 1 implies

$$\begin{aligned}
\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t})\|^2|\mathcal{F}_{t,0}\right] &\leq \frac{2L}{v_{\min}\alpha(k_{\rm in}-1)}\left(\mathrm{W}(\widehat{S}_{t,0}) - \min \mathrm{W}\right) \\
&+ \frac{v_{\max}}{v_{\min}}\frac{k_{\rm in}}{k_{\rm in}-1}\|\mathcal{E}_t\|^2 \;; \quad (8)
\end{aligned}$$

see [28, Corollary 11]. Hence, when $\|\mathcal{E}_t\| = 0$ and W satisfies a Polyak-Łojasiewicz condition [29], i.e.

$$\exists \tau > 0, \forall s \in \mathbb{R}^q, \qquad \mathrm{W}(s) - \min \mathrm{W} \leq \tau\|\nabla \mathrm{W}(s)\|^2 \tag{9}$$

then (8) yields

$$\mathcal{H}_t \overset{\rm def}{=} \overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t})\|^2|\mathcal{F}_{t,0}\right] \leq \frac{2L\tau v_{\max}^2}{v_{\min}\alpha(k_{\rm in}-1)}\|h(\widehat{S}_{t-1,\Xi_{t-1}})\|^2 ,$$

thus establishing a linear rate of the algorithm along the path $\{\widehat{S}_{t,\Xi_t}, t \in [k_{\rm out}]^\star\}$ as soon as $k_{\rm in}$ is large enough:

$$\mathbb{E}\left[\mathcal{H}_t\right] \leq \left(\frac{2L\tau v_{\max}^2}{v_{\min}\alpha(k_{\rm in}-1)}\right)^t \|h(\widehat{S}_{\rm init})\|^2 .$$

Even if the Polyak-Łojasiewicz condition (9) is quite restrictive, the above discussion gives the intuition of the *lock-in* phenomenon which often happens at convergence: a linear rate of convergence is observed when the path is trapped in a neighborhood of its limiting point, which may be the consequence that locally, the Polyak-Łojasiewicz condition holds (see figure 1 in Section 4).

Complexity for ϵ-approximate stationarity. From Theorem 1, Eq. (7) and $\widehat{S}_{t,\Xi_t} = \widehat{S}_{t+1,0}$ (here $\gamma_{t,0} = 0$ and $\mathcal{E}_t = 0$), it holds

$$\frac{v_{\min}\alpha(k_{\rm in}-1)}{2L}\mathbb{E}\left[\mathcal{H}_t\right] \leq \mathbb{E}\left[\mathrm{W}(\widehat{S}_{t,0}) - \mathrm{W}(\widehat{S}_{t+1,0})\right] .$$

Therefore,

$$\frac{1}{k_{\rm out}}\sum_{t=1}^{k_{\rm out}}\mathbb{E}\left[\mathcal{H}_t\right] \leq \frac{2L\left(\mathrm{W}(\widehat{S}_{\rm init}) - \min \mathrm{W}\right)}{v_{\min}\alpha(k_{\rm in}-1)k_{\rm out}} . \tag{10}$$

Eq. (10) establishes that in order to obtain an ϵ-approximate stationary point, it is sufficient to stop the algorithm at the end of the epoch $\#T$, where T is sampled uniformly from $[k_{\rm out}]^\star$ with $k_{\rm out} = O(L/(\epsilon\alpha k_{\rm in}))$ - and return $\widehat{S}_{T,\Xi_T}$. To do such, the mean number of conditional expectations evaluations is $\mathcal{K}_{\rm CE} \overset{\rm def}{=} n + nk_{\rm out} +$

$2\mathsf{b}k_{\rm in}k_{\rm out}$; and the mean number of optimization steps is $\mathcal{K}_{\rm Opt} \stackrel{\rm def}{=} k_{\rm out} + k_{\rm in}k_{\rm out}$. By choosing $k_{\rm in} = O(\sqrt{n})$ and $k_{\rm in}/\mathsf{b} = O(1)$, we have $\mathcal{K}_{\rm CE} = O(n + L\sqrt{n}/(\epsilon\alpha))$ and $\mathcal{K}_{\rm Opt} = O(L/(\epsilon\alpha))$. Similar randomized terminating strategies were proposed in the literature: their optimal complexity in terms of conditional expectations evaluations is $O(\epsilon^{-2})$ for `Online-EM` [15], $O(\epsilon^{-1}n)$ for `i-EM` [14], $O(\epsilon^{-1}n^{2/3})$ for `sEM-vr` [16,17], $O(\{\epsilon^{-1}n^{2/3}\} \wedge \{\epsilon^{-3/2}\sqrt{n}\})$ for `FIEM` [17,22] and $O(\epsilon^{-1}\sqrt{n})$ for `SPIDER-EM` - see [18, section 6] for a comparison of the complexities $\mathcal{K}_{\rm CE}$ and $\mathcal{K}_{\rm opt}$ of these incremental EM algorithms. Hence, `Geom-SPIDER-EM` has the same complexity bounds as `SPIDER-EM`, and they are optimal among the class of incremental EM algorithms.

4. NUMERICAL ILLUSTRATION

We perform experiments on the MNIST dataset, which consists of $n = 6 \times 10^4$ images of handwritten digits, each with 784 pixels. We pre-process the datas as detailed in [22, section 5]: 67 uninformative pixels are removed from each image, then a principal component analysis is applied to further reduce the dimension; we keep the 20 principal components of each observation. The learning problem consists in fitting a Gaussian mixture model with $g = 12$ components having the same covariance matrix: θ collects the weights of the mixture, the expectations of the components (i.e. g vectors in $\mathbb{R}^{20}$) and a 20×20 covariance matrix; F is the negative normalized log-likelihood (no penalty term). All the algorithms start from $\widehat{S}_{\rm init} = \bar{s} \circ \mathsf{T}(\theta_{\rm init})$ such that $-F(\theta_{\rm init}) = -58.3$, and their first two epochs are `Online-EM` iterations. The first epoch with a variance reduction technique is the 3rd; on Fig. 1, the plot starts at epoch $\#2$.

`Geom-SPIDER-EM` is run with a constant step size $\gamma_{t,k} = 0.01$ (and $\gamma_{t,0} = 0$); $k_{\rm out} = 148$ epochs (which are preceded with 2 epochs of `Online-EM`); a mini batch size $\mathsf{b} = \sqrt{n}$. Different strategies are considered for the initialization $\mathsf{S}_{t,0}$ and the parameter of the geometric r.v. Ξ_t. In `full-geom`, $k_{\rm in} = \sqrt{n}/2$ so that the mean total number of conditional expectations evaluations per outer loop is $2\mathsf{b}k_{\rm in} = n$; and $\mathcal{E}_t = 0$ which means that $\mathsf{S}_{t,0}$ requires the computation of the full sum $\bar{s}$ over n terms. In `half-geom`, $k_{\rm in}$ is defined as in `full-geom`, but for all $t \in [k_{\rm out}]^\star$, $\mathsf{S}_{t,0} = (2/n)\sum_{i\in\tilde{\mathcal{B}}_{t,0}} \bar{s}_i \circ \mathsf{T}(\widehat{S}_{t,-1})$ where $\mathcal{B}_{t,0}$ is of cardinality $n/2$; therefore $\mathcal{E}_t \neq 0$. In `quad-geom`, a quadratic growth is considered both for the mean of the geometric random variables: $\mathbb{E}[\Xi_t] = \min(n, \max(20t^2, n/50))/(2\mathsf{b})$; and for the size of the mini batch when computing $\mathsf{S}_{t,0}$: $\mathsf{S}_{t,0} = \tilde{\mathsf{b}}_t^{-1}\sum_{i\in\tilde{\mathcal{B}}_t} \bar{s}_i \circ \mathsf{T}(\widehat{S}_{t,-1})$ with $\tilde{\mathsf{b}}_t = \min(n, \max(20t^2, n/50))$. The `g-SPIDER-EM` with a constant number of inner loops $\xi_t = k_{\rm in} = n/(2\mathsf{b})$ is also run for comparison: different strategies for $\mathsf{S}_{t,0}$ are considered, the same as above (it corresponds to `full-ctt`, `half-ctt` and `quad-ctt` on the plots). Finally, in order to illustrate the benefit of the variance reduction, a pure `Online-EM` is run for 150 epochs, one epoch corresponding to $\sqrt{n}$ updates of the statistics $\widehat{S}$, each of them requiring a mini batch $\mathcal{B}_{k+1}$ of size $\sqrt{n}$ (see Eq.(6)).

The algorithms are compared through an estimation of the quantile of order 0.5 of $\|h(\widehat{S}_{t,\Xi_t})\|^2$ over 30 independent realizations. It is plotted versus the number of epochs t in Fig. 1 and the number of conditional expectations (CE) evaluations in Fig. 2. They are also compared through the objective function F along the path; the mean value over 30 independent paths is displayed versus the number of CE, see Fig. 3.

We first observe that `Online-EM` has a poor convergence rate, thus justifying the interest of variance reduction techniques as shown in Fig. 1. Having a persistent bias along iterations when defining $\mathsf{S}_{t,0}$

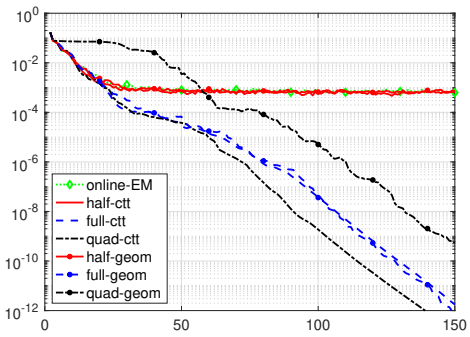

Fig. 1. Quantile 0.5 of $\|h(\widehat{S}_{t,\Xi_t})\|^2$ vs the number of epochs

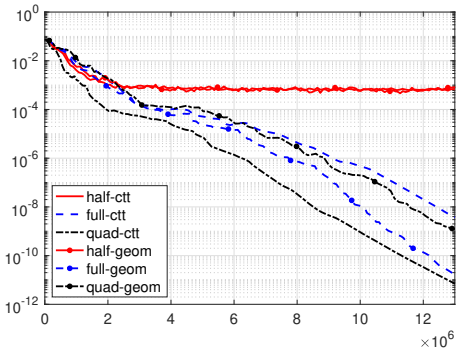

Fig. 2. Quantile 0.5 of $\|h(\widehat{S}_{t,\Xi_t})\|^2$ vs the number of CE evaluations

i.e. considering $\tilde{\mathsf{b}}_t \neq n$ and therefore $\mathcal{E}_t \neq 0$, is also a bad strategy as seen in Fig. 1, 2 for `half-ctt` and `half-geom`. For the four other `g-SPIDER-EM` strategies, we observe a linear convergence rate in Fig. 1, 2. The best strategy, both in terms of CE evaluations and in terms of efficiency given a number of epochs, is `quad-ctt`: a constant and deterministic number of inner loops ξ_t combined with an increasing accuracy when computing $\mathsf{S}_{t,0}$; therefore, during the first iterations, it is better to reduce the computational cost of the algorithm by considering $\tilde{\mathsf{b}}_t \ll n$. When $\mathcal{E}_t = 0$ (i.e. $\tilde{\mathsf{b}}_t = n$ so the computational cost of $\mathsf{S}_{t,0}$ is maximal), it is possible to reduce the total CE computational cost of the algorithm by considering a random number of inner loops (see `full-geom` and `full-ctt` on Fig. 1, 2). Finally, the strategy which consists in increasing both $\tilde{\mathsf{b}}_t$ and the number of inner loops, does not look the best one (see `quad-ctt` and `quad-geom` on Fig. 1 to Fig. 3).

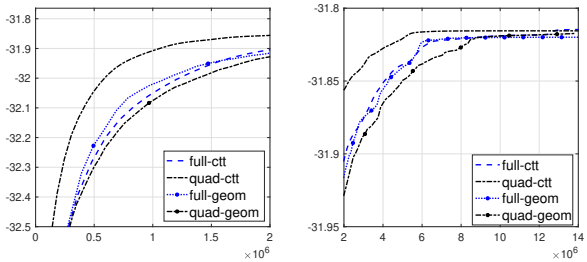

Fig. 3. (Left) $-F$ vs CE, until $2e6$. (Right) $-F$ vs CE, after $2e6$.

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SUPPLEMENTARY MATERIAL, PAPER "GEOM-SPIDER-EM: FASTER VARIANCE REDUCED STOCHASTIC EXPECTATION MAXIMIZATION FOR NONCONVEX FINITE-SUM OPTIMIZATION"

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1. PROOF OF THEOREM 1

Let $\{\mathcal{E}_t, t \in [k_{\rm out}]^\star\}$ and $\{\mathcal{B}_{t,k+1}, t \in [k_{\rm out}]^\star, k \in [\xi_t - 1]\}$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define the filtrations $\mathcal{F}_{1,0} \stackrel{\rm def}{=} \sigma(\mathcal{E}_1)$, $\mathcal{F}_{t,0} \stackrel{\rm def}{=} \sigma(\mathcal{F}_{t-1,\xi_t}, \mathcal{E}_t)$ for $t \geq 2$, and $\mathcal{F}_{t,k+1} \stackrel{\rm def}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$ for $t \in [k_{\rm out}]^\star, k \in [\xi_t - 1]$.

For $\rho_t \in (0,1)$, set

$$\overline{\mathbb{E}}_t[\phi(\Xi_t)|\mathcal{F}_{t,0}] \stackrel{\rm def}{=} (1-\rho_t) \sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k)|\mathcal{F}_{t,0}] \ ,$$

for any measurable positive function ϕ.

A1. *$\Theta \subseteq \mathbb{R}^d$ is a convex set. $(\mathsf{Z}, \mathcal{Z})$ is a measurable space and μ is a σ-finite positive measure on $\mathcal{Z}$. The functions $\mathsf{R} : \Theta \to \mathbb{R}$, $\phi : \Theta \to \mathbb{R}^q$, $s_i : \mathsf{Z} \to \mathbb{R}^q$, $h_i : \mathsf{Z} \to \mathbb{R}_+$ for all $i \in [n]^\star$ are measurable. For any $\theta \in \Theta$ and $i \in [n]^\star$, $|\mathcal{L}_i(\theta)| < \infty$.*

A2. *The expectations $\bar{s}_i(\theta)$ are well defined for all $\theta \in \Theta$ and $i \in [n]^\star$. For any $s \in \mathbb{R}^q$, $\mathrm{Argmin}_{\theta \in \Theta} \ (-\langle s, \phi(\theta) \rangle + \mathsf{R}(\theta))$ is a (non empty) singleton denoted by $\{\mathsf{T}(s)\}$.*

A 3. *The functions ϕ and R are continuously differentiable on Θ^v, where Θ^v is an open neighborhood of Θ when Θ is not open and $\Theta^v \stackrel{\rm def}{=} \Theta$ otherwise. T is continuously differentiable on $\mathbb{R}^q$. The function F is continuously differentiable on Θ^v and for any $\theta \in \Theta$, $\nabla F(\theta) = -\nabla\phi(\theta)^T \bar{s}(\theta) + \nabla \mathsf{R}(\theta)$. For any $s \in \mathbb{R}^q$, $B(s) \stackrel{\rm def}{=} \nabla(\phi \circ \mathsf{T})(s)$ is a symmetric $q \times q$ matrix and there exist $0 < v_{min} \leq v_{max} < \infty$ such that for all $s \in \mathbb{R}^q$, the spectrum of $B(s)$ is in $[v_{min}, v_{max}]$. For any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i. The function $s \mapsto \nabla(F \circ \mathsf{T})(s) = -B(s)\,(\bar{s} \circ T(s) - s)$ is globally Lipschitz on $\mathbb{R}^q$ with constant $L_{\dot{\rm W}}$.*

Lemma 1. *Let $\rho \in (0,1)$ and $\{D_k, k \geq 0\}$ be real numbers such that $\sum_{k \geq 0} \rho^k |D_k| < \infty$. Let $\xi \sim \mathcal{G}^\star(1-\rho)$. Then $\mathbb{E}[D_{\xi-1}] = \rho\mathbb{E}[D_\xi] + (1-\rho)D_0 = \mathbb{E}[D_\xi] + (1-\rho)(D_0 - \mathbb{E}[D_\xi])$.*

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Proof. By definition of ξ,

$$\begin{aligned}
\mathbb{E}\left[D_\xi\right] &= (1-\rho) \sum_{k \geq 1} \rho^{k-1} D_k \\
&= \rho^{-1}(1-\rho) \sum_{k \geq 1} \rho^k D_k \\
&= \rho^{-1}(1-\rho) \sum_{k \geq 0} \rho^k D_k - \rho^{-1}(1-\rho) D_0 \\
&= \rho^{-1}(1-\rho) \sum_{k \geq 1} \rho^{k-1} D_{k-1} - \rho^{-1}(1-\rho) D_0 \\
&= \rho^{-1}\mathbb{E}\left[D_{\xi-1}\right] - \rho^{-1}(1-\rho) D_0 \ .
\end{aligned}$$

This yields $\rho\mathbb{E}\left[D_\xi\right] = \mathbb{E}\left[D_{\xi-1}\right] - (1-\rho)D_0$ and concludes the proof. □

Lemma 2. *For any $t \in [k_{\rm out}]^\star$, $k \in [\xi_t]^\star$, $\mathcal{B}_{t,k}$ and $\mathcal{F}_{t,k-1}$ are independent. In addition, for any $s \in \mathbb{R}^q$, $\mathsf{b}^{-1}\mathbb{E}\left[\sum_{i \in \mathcal{B}_{t,k}} \bar{s}_i \circ \mathsf{T}(s)\right] = \bar{s} \circ \mathsf{T}(s)$. Finally, assume that for any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i. Then for any $s, s' \in \mathbb{R}^q$,*

$$\begin{aligned}
\mathbb{E}\left[\left\|\mathsf{b}^{-1} \sum_{i \in \mathcal{B}_{t,k}} \left\{\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s')\right\} - \bar{s} \circ \mathsf{T}(s) + \bar{s} \circ \mathsf{T}(s')\right\|^2\right] \\
\leq \frac{1}{\mathsf{b}} \left(L^2 \|s - s'\|^2 - \|\bar{s} \circ \mathsf{T}(s) - \bar{s} \circ \mathsf{T}(s')\|^2\right) \ ,
\end{aligned}$$

where $L^2 \stackrel{\rm def}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Proof. See [1, Lemma 4]; the proof holds true when $\mathcal{B}_{t,k}$ is sampled with or without replacement. □

Proposition 3. *For any $t \in [k_{\rm out}]^\star$, $k \in [\xi_t - 1]$,*

$$\mathbb{E}\left[\mathsf{S}_{t,k+1} | \mathcal{F}_{t,k}\right] - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) = \mathsf{S}_{t,k} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1}) \ ,$$

and

$$\mathbb{E}\left[\mathsf{S}_{t,k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) | \mathcal{F}_{t,0}\right] = \mathcal{E}_t \ .$$

Proof. Let $t \in [k_{\rm out}]^\star$, $k \in [\xi_t - 1]$. By Lemma 2,

$$\mathbb{E}\left[\mathsf{S}_{t,k+1} | \mathcal{F}_{t,k}\right] = \mathsf{S}_{t,k} + \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k}) - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1}) \ .$$

By definition of $\mathsf{S}_{t,0}$ and of the filtrations, $\mathsf{S}_{t,0} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,-1}) = \mathcal{E}_t \in \mathcal{F}_{t,0}$. The proof follows by induction on k. □

Proposition 4. *Assume that for any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i. For any $t \in [k_{\rm out}]^\star$, $k \in [\xi_t - 1]$,*

$$\begin{aligned}
&\mathbb{E}\left[\|\mathsf{S}_{t,k+1} - \mathbb{E}\left[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}\right]\|^2|\mathcal{F}_{t,k}\right] \\
&\le \frac{1}{\mathsf{b}}\left(L^2\|\widehat{S}_{t,k} - \widehat{S}_{t,k-1}\|^2 - \|\bar{s}\circ\mathsf{T}(\widehat{S}_{t,k}) - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k-1})\|^2\right) \\
&\le \frac{L^2}{\mathsf{b}}\gamma_{t,k}^2\|\mathsf{S}_{t,k} - \widehat{S}_{t,k-1}\|^2 \ ,
\end{aligned}$$

where $L^2 \stackrel{\rm def}{=} n^{-1}\sum_{i=1}^n L_i^2$. By convention, $\gamma_{1,0} = 0$.

Proof. Let $t \in [k_{\rm out}]^\star$, $k \in [\xi_t - 1]$. By Lemma 2, Proposition 3, the definition of $\mathsf{S}_{t,k+1}$ and of the filtration $\mathcal{F}_{t,k}$,

$$\begin{aligned}
&\mathsf{S}_{t,k+1} - \mathbb{E}\left[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}\right] \\
&= \mathsf{S}_{t,k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k}) - \mathsf{S}_{t,k} + \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k-1}) \\
&= \mathsf{b}^{-1}\sum_{i\in\mathcal{B}_{t,k+1}}\{\bar{s}_i\circ\mathsf{T}(\widehat{S}_{t,k}) - \bar{s}_i\circ\mathsf{T}(\widehat{S}_{t,k-1})\} \\
&\quad - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k}) + \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k-1}) \ .
\end{aligned}$$

We then conclude by Lemma 2 for the first inequality; and by using the definition of $\widehat{S}_{t,k}$ for the second one:

$$\widehat{S}_{t,k} - \widehat{S}_{t,k-1} = \gamma_{t,k}\left(\mathsf{S}_{t,k} - \widehat{S}_{t,k-1}\right)$$

except when $t = 1$ and $k = 0$, where $\widehat{S}_{1,0} - \widehat{S}_{1,-1} = 0$. □

Proposition 5. *Assume that for any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i. For any $t \in [k_{\rm out}]^\star$, $k \in [\xi_t - 1]$,*

$$\begin{aligned}
&\mathbb{E}\left[\|\mathsf{S}_{t,k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k})\|^2|\mathcal{F}_{t,k}\right] \\
&\le \frac{L^2}{\mathsf{b}}\gamma_{t,k}^2\|\mathsf{S}_{t,k} - \widehat{S}_{t,k-1}\|^2 + \|\mathsf{S}_{t,k} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k-1})\|^2 \ ,
\end{aligned}$$

where $L^2 \stackrel{\rm def}{=} n^{-1}\sum_{i=1}^n L_i^2$. By convention, $\gamma_{1,0} = 0$.

Proof. By definition of the conditional expectation, we have for any r.v. $\phi(V)$

$$\mathbb{E}\left[\|U - \phi(V)\|^2|V\right] = \mathbb{E}\left[\|U - \mathbb{E}\left[U|V\right]\|^2|V\right] + \|\mathbb{E}\left[U|V\right] - \phi(V)\|^2 \ .$$

The proof follows from this equality and Propositions 3 and 4. □

Corollary 6. *Assume that for any $i \in [n]^\star$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz with constant L_i. For any $t \in [k_{\rm out}]^\star$, let $\rho_t \in (0,1)$ and $\Xi_t \sim \mathcal{G}^\star(1-\rho_t)$. For any $t \in [k_{\rm out}]^\star$,*

$$\begin{aligned}
&\overline{\mathbb{E}}_t\left[\left(\gamma_{t,\Xi_t} - \rho_t\gamma_{t,\Xi_t+1}\right)\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \\
&\le \frac{L^2\rho_t}{\mathsf{b}}\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t+1}\gamma_{t,\Xi_t}^2\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \\
&\quad + \frac{L^2(1-\rho_t)}{\mathsf{b}}\gamma_{t,1}\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2 + (1-\rho_t)\gamma_{t,1}\|\mathcal{E}_t\|^2 \ ,
\end{aligned}$$

where $L^2 \stackrel{\rm def}{=} n^{-1}\sum_{i=1}^n L_i^2$.

Proof. Let $t \in [k_{\rm out}]^\star$ and $k \in [\xi_t - 1]$. From Proposition 5 and since $\mathcal{F}_{t,0} \subseteq \mathcal{F}_{t,k}$ for $k \in [\xi_t - 1]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\|\mathsf{S}_{t,k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k})\|^2|\mathcal{F}_{t,0}\right] \\
&\le \mathbb{E}\left[\|\mathsf{S}_{t,k} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k-1})\|^2 + \frac{L^2}{\mathsf{b}}\gamma_{t,k}^2\|\mathsf{S}_{t,k} - \widehat{S}_{t,k-1}\|^2|\mathcal{F}_{t,0}\right] \ .
\end{aligned}$$

Multiply by $\gamma_{t,k+1}$ and apply with $k = \xi_t - 1$:

$$\begin{aligned}
&\mathbb{E}\left[\gamma_{t,\xi_t}\|\mathsf{S}_{t,\xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \\
&\le \mathbb{E}\left[\gamma_{t,\xi_t}\|\mathsf{S}_{t,\xi_t-1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\xi_t-2})\|^2|\mathcal{F}_{t,0}\right] \\
&\quad + \frac{L^2}{\mathsf{b}}\mathbb{E}\left[\gamma_{t,\xi_t}\gamma_{t,\xi_t-1}^2\|\mathsf{S}_{t,\xi_t-1} - \widehat{S}_{t,\xi_t-2}\|^2|\mathcal{F}_{t,0}\right] \ .
\end{aligned}$$

This implies

$$\begin{aligned}
&\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \\
&\le \overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\|\mathsf{S}_{t,\Xi_t-1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-2})\|^2|\mathcal{F}_{t,0}\right] \\
&\quad + \frac{L^2}{\mathsf{b}}\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\gamma_{t,\Xi_t-1}^2\|\mathsf{S}_{t,\Xi_t-1} - \widehat{S}_{t,\Xi_t-2}\|^2|\mathcal{F}_{t,0}\right] \ .
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\|\mathsf{S}_{t,\Xi_t-1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-2})\|^2|\mathcal{F}_{t,0}\right] \\
&= \rho_t\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t+1}\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \\
&\quad + (1-\rho_t)\gamma_{t,1}\|\mathsf{S}_{t,0} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,-1})\|^2 \ ;
\end{aligned}$$

by definition of $\mathsf{S}_{t,0}$ and $\mathcal{F}_{t,0}$, the last term is equal to $(1-\rho_t)\gamma_{t,1}\|\mathcal{E}_t\|^2$. By Lemma 1, we have

$$\begin{aligned}
&\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\gamma_{t,\Xi_t-1}^2\|\mathsf{S}_{t,\Xi_t-1} - \widehat{S}_{t,\Xi_t-2}\|^2|\mathcal{F}_{t,0}\right] \\
&= \rho_t\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t+1}\gamma_{t,\Xi_t}^2\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \\
&\quad + (1-\rho_t)\gamma_{t,1}\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2 \ .
\end{aligned}$$

This concludes the proof. □

Lemma 7. *For any $h, s, S \in \mathbb{R}^q$ and any $q \times q$ symmetric matrix B, it holds*

$$-2\left\langle Bh, S\right\rangle = -\left\langle BS, S\right\rangle - \left\langle Bh, h\right\rangle + \left\langle B\{h - S\}, h - S\right\rangle \ .$$

Proposition 8. *Assume A1 to A3. For any $t \in [k_{\rm out}]^\star$ and $k \in [\xi_t - 1]$,*

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{W}(\widehat{S}_{t,k+1})|\mathcal{F}_{t,0}\right] + \frac{v_{\min}}{2}\gamma_{t,k+1}\mathbb{E}\left[\|h(\widehat{S}_{t,k})\|^2|\mathcal{F}_{t,0}\right] \\
&\le \mathbb{E}\left[\mathrm{W}(\widehat{S}_{t,k})|\mathcal{F}_{t,0}\right] \\
&\quad + \frac{v_{\max}}{2}\gamma_{t,k+1}\mathbb{E}\left[\|\mathsf{S}_{t,k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,k})\|^2|\mathcal{F}_{t,0}\right] \\
&\quad - \frac{\gamma_{t,k+1}}{2}\left(v_{\min} - \gamma_{t,k+1}L_{\dot{\mathrm{W}}}\right)\mathbb{E}\left[\|\mathsf{S}_{t,k+1} - \widehat{S}_{t,k}\|^2|\mathcal{F}_{t,0}\right] \ .
\end{aligned}$$

Proof. Since W is continuously differentiable with $L_{\dot{\mathrm{W}}}$-Lipschitz gradient, then for any $s, s' \in \mathbb{R}^q$,

$$\mathrm{W}(s') - \mathrm{W}(s) \le \left\langle\nabla\,\mathrm{W}(s), s' - s\right\rangle + \frac{L_{\dot{\mathrm{W}}}}{2}\|s' - s\|^2 \ .$$

Set $s' = s + \gamma S$ where $\gamma > 0$ and $S \in \mathbb{R}^q$. Since $\nabla\,\mathrm{W}(s) = -B(s)h(s)$ and $B(s)$ is symmetric, apply Lemma 7 with $h \leftarrow h(s)$, $B \leftarrow B(s)$ and $S = (s' - s)/\gamma$; this yields

$$\begin{aligned}
&\mathrm{W}(s + \gamma S) - \mathrm{W}(s) \le -\frac{\gamma}{2}\left\langle B(s)S, S\right\rangle - \frac{\gamma}{2}\left\langle B(s)h(s), h(s)\right\rangle \\
&\quad + \frac{\gamma}{2}\left\langle B(s)\{h(s) - S\}, h(s) - S\right\rangle + \frac{L_{\dot{\mathrm{W}}}}{2}\gamma^2\|S\|^2 \ .
\end{aligned}$$

Since $\|a\|^2 v_{\min} \leq \langle B(s)a, a\rangle \leq v_{\max}\|a\|^2$ for any $a \in \mathbb{R}^q$, we have

$$\mathrm{W}(s+\gamma S) - \mathrm{W}(s) \leq -\frac{\gamma v_{\min}}{2}\|S\|^2 - \frac{\gamma v_{\min}}{2}\|h(s)\|^2 + \frac{\gamma v_{\max}}{2}\|h(s)-S\|^2 + \frac{L_{\dot{\mathrm{W}}}}{2}\gamma^2\|S\|^2 \ .$$

Let $t \in [k_{\mathrm{out}}]^\star$ and $k \in [\xi_t - 1]$. Applying this inequality with $s \leftarrow \widehat{S}_{t,k}$, $\gamma \leftarrow \gamma_{t,k+1}$, $S \leftarrow \mathsf{S}_{t,k+1} - \widehat{S}_{t,k}$ (which yields $s + \gamma S = \widehat{S}_{t,k+1}$), and then the conditional expectation yield the result. □

Proposition 9. *Assume A1 to A3. For any $t \in [k_{\mathrm{out}}]^\star$*

$$\begin{aligned}&\mathrm{W}(\widehat{S}_{t+1,0}) - \mathrm{W}(\widehat{S}_{t+1,-1})\\ &\leq -\frac{\gamma_{t+1,0} v_{\min}}{2}\|h(\widehat{S}_{t+1,-1})\|^2 + \frac{v_{\max}\gamma_{t+1,0}}{2}\|\mathcal{E}_{t+1}\|^2\\ &- \frac{\gamma_{t+1,0}}{2}\left(v_{\min} - \gamma_{t+1,0}L_{\dot{\mathrm{W}}}\right)\|\mathsf{S}_{t+1,0} - \widehat{S}_{t+1,-1}\|^2 \ .\end{aligned}$$

Proof. As in the proof of Proposition 8, we write for any $s, s' \in \mathbb{R}^q$,

$$\mathrm{W}(s') - \mathrm{W}(s) \leq \left\langle \nabla \mathrm{W}(s), s' - s\right\rangle + \frac{L_{\dot{\mathrm{W}}}}{2}\|s'-s\|^2 \ .$$

With Lemma 7, this yields when $s' = s + \gamma S$ for $\gamma > 0$ and $S \in \mathbb{R}^q$

$$\mathrm{W}(s+\gamma S) - \mathrm{W}(s) \leq -\frac{\gamma}{2}\left(v_{\min} - \gamma L_{\dot{\mathrm{W}}}\right)\|S\|^2 - \frac{\gamma v_{\min}}{2}\|h(s)\|^2 + \frac{v_{\max}\gamma}{2}\|h(s)-S\|^2 \ .$$

Apply this inequality with $\gamma \leftarrow \gamma_{t+1,0}$, $s \leftarrow \widehat{S}_{t+1,-1}$ and $S \leftarrow \mathsf{S}_{t+1,0} - \widehat{S}_{t+1,-1}$. This yields $s + \gamma S = \widehat{S}_{t+1,0}$ and

$$h(s) - S = \bar{s} \circ \mathsf{T}(\widehat{S}_{t+1,-1}) - \mathsf{S}_{t+1,0} = -\mathcal{E}_{t+1} \ .$$

□

Theorem 10. *Assume A1 to A3. For any $t \in [k_{\mathrm{out}}]^\star$, let $\rho_t \in (0,1)$ and $\Xi_t \sim \mathcal{G}^\star(1-\rho_t)$. Finally, choose $\gamma_{t,k+1} = \gamma_t > 0$ for any $k \geq 0$. For any $t \in [k_{\mathrm{out}}]^\star$,*

$$\begin{aligned}&\frac{v_{\min}\gamma_t}{2(1-\rho_t)}\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \leq \mathrm{W}(\widehat{S}_{t,0}) - \overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}}{2(1-\rho_t)}\frac{L^2}{\mathsf{b}}\gamma_t\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2(1-\rho_t)}\gamma_t\|\mathcal{E}_t\|^2\\ &- \frac{\gamma_t}{2(1-\rho_t)}\left(v_{\min} - \gamma_t L_{\dot{\mathrm{W}}} - \frac{v_{\max}L^2\rho_t}{(1-\rho_t)\mathsf{b}}\gamma_t^2\right)\cdots\\ &\quad\times\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \ .\end{aligned}$$

By convention, $\gamma_{1,0} = 0$.

Proof. Apply Proposition 8 with $k \leftarrow \xi_t - 1$ and then set $\xi_t \leftarrow \Xi_t$; this yields

$$\begin{aligned}&\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right] + \frac{v_{\min}}{2}\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\|h(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &\leq \overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t-1})|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}}{2}\overline{\mathbb{E}}_t\left[\gamma_{t,\Xi_t}\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &- \overline{\mathbb{E}}_t\left[\frac{\gamma_{t,\Xi_t}}{2}\left(v_{\min} - \gamma_{t,\Xi_t}L_{\dot{\mathrm{W}}}\right)\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \ .\end{aligned}$$

Since $\Xi_t \geq 1$ and $\gamma_{t,k} = \gamma_t$ for any $k \geq 1$, we have

$$\begin{aligned}&\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right] + \frac{v_{\min}}{2}\gamma_t\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &\leq \overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t-1})|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}}{2}\gamma_t\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &- \frac{\gamma_t}{2}\left(v_{\min} - \gamma_t L_{\dot{\mathrm{W}}}\right)\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \ .\end{aligned}$$

By Lemma 1, it holds

$$\begin{aligned}&\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right] = \overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t-1})|\mathcal{F}_{t,0}\right]\\ &+ (1-\rho_t)\left(\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right] - \mathrm{W}(\widehat{S}_{t,0})\right) \ .\end{aligned}$$

Furthermore, by Corollary 6 applied with $\gamma_{t,\Xi_t} = \gamma_{t,\Xi_t+1} = \gamma_t$

$$\begin{aligned}&(1-\rho_t)\gamma_t\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \bar{s}\circ\mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &\leq \frac{L^2\rho_t}{\mathsf{b}}\gamma_t^3\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right]\\ &+ \frac{L^2(1-\rho_t)}{\mathsf{b}}\gamma_t\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2 + (1-\rho_t)\gamma_t\|\mathcal{E}_t\|^2 \ ,\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{v_{\min}}{2}\gamma_t\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right]\\ &\leq (1-\rho_t)\,\mathrm{W}(\widehat{S}_{t,0}) - (1-\rho_t)\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}}{2}\frac{L^2\rho_t}{(1-\rho_t)\mathsf{b}}\gamma_t^3\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}}{2}\frac{L^2}{\mathsf{b}}\gamma_t\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2}\gamma_t\|\mathcal{E}_t\|^2\\ &- \frac{\gamma_t}{2}\left(v_{\min} - \gamma_t L_{\dot{\mathrm{W}}}\right)\overline{\mathbb{E}}_t\left[\|\mathsf{S}_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2|\mathcal{F}_{t,0}\right] \ .\end{aligned}$$

This concludes the proof. □

Corollary 11 (of Theorem 10)**.** *Assume that for any $t \in [k_{\mathrm{out}}]^\star$, $1-\rho_t = 1/k_{\mathrm{in}}$ and $\gamma_t = \alpha/L$ where $\alpha > 0$ satisfies*

$$v_{\min} - \alpha\frac{L_{\dot{\mathrm{W}}}}{L} - \alpha^2 v_{\max}\frac{k_{\mathrm{in}}}{\mathsf{b}}\left(1 - \frac{1}{k_{\mathrm{in}}}\right) > 0 \ .$$

For any $t \in [k_{\mathrm{out}}]^\star$,

$$\begin{aligned}&\left(\frac{\alpha(k_{\mathrm{in}}-1)}{L} + \gamma_{t+1,0}\right)\frac{v_{\min}}{2}\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t})\|^2|\mathcal{F}_{t,0}\right]\\ &\leq \mathrm{W}(\widehat{S}_{t,0}) - \mathbb{E}\left[\mathrm{W}(\widehat{S}_{t+1,0})|\mathcal{F}_{t,0}\right]\\ &- \frac{\gamma_{t+1,0}}{2}\left(v_{\min} - \gamma_{t+1,0}L_{\dot{\mathrm{W}}}\right)\mathbb{E}\left[\|\mathsf{S}_{t+1,0} - \widehat{S}_{t+1,-1}\|^2|\mathcal{F}_{t,0}\right]\\ &+ \frac{v_{\max}k_{\mathrm{in}}}{2}\frac{\alpha L}{\mathsf{b}}\gamma_{t,0}^2\|\mathsf{S}_{t,0} - \widehat{S}_{t,-1}\|^2\\ &+ \frac{v_{\max}\alpha k_{\mathrm{in}}}{2L}\|\mathcal{E}_t\|^2 + \frac{v_{\max}\gamma_{t+1,0}}{2}\mathbb{E}\left[\|\mathcal{E}_{t+1}\|^2|\mathcal{F}_{t,0}\right] \ .\end{aligned}$$

Proof. Let $t \in [k_{\rm out}]^\star$. By Proposition 9, since $\widehat{S}_{t,\xi_t} = \widehat{S}_{t+1,-1}$ we have

$$
\begin{aligned}
&- \mathbb{E}\left[\mathrm{W}(\widehat{S}_{t,\xi_t})|\mathcal{F}_{t,0}\right] \leq -\mathbb{E}\left[\mathrm{W}(\widehat{S}_{t+1,0})|\mathcal{F}_{t,0}\right] \\
&- \frac{\gamma_{t+1,0}\upsilon_{\min}}{2}\mathbb{E}\left[\|h(\widehat{S}_{t,\xi_t})\|^2|\mathcal{F}_{t,0}\right] \\
&+ \frac{\upsilon_{\max}\gamma_{t+1,0}}{2}\mathbb{E}\left[\|\mathcal{E}_{t+1}\|^2|\mathcal{F}_{t,0}\right] \\
&- \frac{\gamma_{t+1,0}}{2}\left(\upsilon_{\min} - \gamma_{t+1,0}L_{\dot{\mathrm{W}}}\right)\mathbb{E}\left[\|\mathsf{S}_{t+1,0} - \widehat{S}_{t+1,-1}\|^2|\mathcal{F}_{t,0}\right] \ .
\end{aligned}
$$

The previous inequality remains true when $\mathbb{E}\left[\mathrm{W}(\widehat{S}_{t,\xi_t})|\mathcal{F}_{t,0}\right]$ is replaced with $\overline{\mathbb{E}}_t\left[\mathrm{W}(\widehat{S}_{t,\Xi_t})|\mathcal{F}_{t,0}\right]$; and $\mathbb{E}\left[\|h(\widehat{S}_{t,\xi_t})\|^2|\mathcal{F}_{t,0}\right]$ with $\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t})\|^2|\mathcal{F}_{t,0}\right]$. The proof follows from Theorem 10, and (see Lemma 1)

$$
\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t-1})\|^2|\mathcal{F}_{t,0}\right] \geq \rho_t\overline{\mathbb{E}}_t\left[\|h(\widehat{S}_{t,\Xi_t})\|^2|\mathcal{F}_{t,0}\right] \ .
$$

We also use $\gamma_t = \alpha/L$ and $\rho_t/(1-\rho_t) = k_{\rm in} - 1$. □

2. NUMERICAL ILLUSTRATION

By convention, vectors are column-vectors. For a matrix a, a^T denotes its transpose.

2.1. The model

The observations $(Y_1, \cdots, Y_n)$ are assumed i.i.d. with distribution

$$\sum_{\ell=1}^{g} \alpha_\ell \, \mathcal{N}_p(\mu_\ell, \Sigma)$$

where $\alpha_\ell \geq 0$ and $\sum_{\ell=1}^g \alpha_\ell = 1$. The negative normalized log-likelihood is given by (up to an additive constant)

$$\theta \mapsto F(\theta) \stackrel{\text{def}}{=} -\frac{1}{2}\log\det(\Sigma^{-1}) - \frac{1}{n}\sum_{i=1}^{n} \log \sum_{\ell=1}^{g} \alpha_\ell \exp\left(-\frac{1}{2}(Y_i-\mu_\ell)^T\Sigma^{-1}(Y_i-\mu_\ell)\right).$$

The parameter θ collects the weights, the expectations, and the covariance matrix

$$\theta \stackrel{\text{def}}{=} (\alpha_1, \cdots, \alpha_g, \mu_1, \cdots, \mu_g, \Sigma)$$

so that $\theta \in \Theta$ with

$$\Theta \stackrel{\text{def}}{=} \left\{ \alpha_\ell \geq 0 \text{ for } \ell \in [g]^\star \text{ s.t. } \sum_{\ell=1}^{g} \alpha_\ell = 1 \right\} \times (\mathbb{R}^p)^g \times \mathcal{M}_p^+;$$

$\mathcal{M}_p^+$ denotes the set of the $p \times p$ positive definite matrices.

For all $i \in [n]^\star$, we write

$$\begin{aligned}
&-\log \sum_{\ell=1}^{g} \alpha_\ell \exp\left(-\frac{1}{2}(Y_i-\mu_\ell)^T\Sigma^{-1}(Y_i-\mu_\ell)\right) \\
&= -\log \sum_{\ell=1}^{g} \exp\left(-\frac{1}{2}(Y_i-\mu_\ell)^T\Sigma^{-1}(Y_i-\mu_\ell) + \log\alpha_\ell\right) \\
&= -\log \sum_{z=1}^{g} \exp\left(-\frac{1}{2}\sum_{\ell=1}^{g} \mathsf{1}_{z=\ell}(Y_i-\mu_\ell)^T\Sigma^{-1}(Y_i-\mu_\ell) \right. \\
&\left. + \sum_{\ell=1}^{g} \mathsf{1}_{z=\ell}\log\alpha_\ell\right) \\
&= \frac{1}{2}Y_i^T\Sigma^{-1}Y_i - \log\sum_{z=1}^{g}\exp\left(\sum_{\ell=1}^{g}\mathsf{1}_{z=\ell}Y_i^T\Sigma^{-1}\mu_\ell\right)\cdots \\
&\times \exp\left(-\frac{1}{2}\sum_{\ell=1}^{g}\mathsf{1}_{z=\ell}\left\{\mu_\ell^T\Sigma^{-1}\mu_\ell - 2\log\alpha_\ell\right\}\right)
\end{aligned}$$

Hence, $F(\theta)$ is of the form

$$-\frac{1}{n}\sum_{i=1}^{n}\log\int_{\mathsf{Z}} h_i(z)\ \exp(\langle s_i(z), \phi(\theta)\rangle)\,\mu(\mathrm{d}z) + \mathsf{R}(\theta)$$

by setting: μ the counting measure on $[g]^\star$, $\mathsf{Z} = [g]^\star$, $h_i(z) = 1$, and

$$\mathsf{R}(\theta) \stackrel{\text{def}}{=} \frac{1}{2}\mathrm{Trace}\left(\Sigma^{-1}\frac{1}{n}\sum_{i=1}^{n}Y_iY_i^T\right) - \frac{1}{2}\log\det(\Sigma^{-1}),$$

$$\phi(\theta) \stackrel{\text{def}}{=} \begin{bmatrix} \log\alpha_1 - 0.5\mu_1^T\Sigma^{-1}\mu_1 \\ \cdots \\ \log\alpha_g - 0.5\mu_g^T\Sigma^{-1}\mu_g \\ \Sigma^{-1}\mu_1 \\ \cdots \\ \Sigma^{-1}\mu_g \end{bmatrix}, \quad s_i(z) \stackrel{\text{def}}{=} \begin{bmatrix} \mathsf{1}_{z=1} \\ \cdots \\ \mathsf{1}_{z=g} \\ \mathsf{1}_{z=1}Y_i \\ \cdots \\ \mathsf{1}_{z=g}Y_i \end{bmatrix}.$$

Note that $\phi(\theta) \in \mathbb{R}^{g+pg}$ and $s_i(z) \in \mathbb{R}^{g+pg}$.

From these expressions, for any $i \in [n]^\star$ and $\theta \in \Theta$, the distribution $z \mapsto p_i(z;\theta)$ is the distribution on $[g]^\star$ given by

$$p_i(\ell;\theta) = \frac{\alpha_\ell \exp\left(-\frac{1}{2}(Y_i-\mu_\ell)^T\Sigma^{-1}(Y_i-\mu_\ell)\right)}{\sum_{u=1}^{g}\alpha_u\exp\left(-\frac{1}{2}(Y_i-\mu_u)^T\Sigma^{-1}(Y_i-\mu_u)\right)}$$

for any $\ell \in [g]^\star$. Therefore

$$\bar{s}_i(\theta) = \begin{bmatrix} p_i(1;\theta) \\ \cdots \\ p_i(g;\theta) \\ p_i(1;\theta)Y_i \\ \cdots \\ p_i(g;\theta)Y_i \end{bmatrix}.$$

For a vector $s \in \mathbb{R}^{g+pg}$, we write

$$s = \begin{bmatrix} s^{(1),1} \\ \cdots \\ s^{(1),g} \\ s^{(2),1} \\ \cdots \\ s^{(2),g} \end{bmatrix},$$

where $s^{(1),\ell} \in \mathbb{R}$ and $s^{(2),\ell} \in \mathbb{R}^p$; with these notations, $\mathsf{T}(s) = (\alpha_1, \cdots, \alpha_g, \mu_1, \cdots, \mu_g, \Sigma)$ is defined by

$$\begin{aligned}
\alpha_\ell &\stackrel{\text{def}}{=} \frac{s^{(1),\ell}}{\sum_{u=1}^{g} s^{(1),u}} \\
\mu_\ell &\stackrel{\text{def}}{=} \frac{s^{(2),\ell}}{s^{(1),\ell}} \\
\Sigma &\stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n}Y_iY_i^T - \sum_{\ell=1}^{g} s^{(1),\ell}\mu_\ell\mu_\ell^T.
\end{aligned}$$

2.2. Additional plots

In the numerical applications, $p = 20$, $g = 12$ and $n = 6 \times 10^4$.

Figure 1 displays the number of conditional expectation evaluations per epoch (top) or cumulated vs the number of epoch (bottom). For `quad-geom`, the total number of conditional expectations (CE) evaluations is

- $\lfloor(\min(n, \max(n/100, 6t^2)))\rfloor$ for the computation of S_t
- and the 2b multiplied by the mean number of ξ_t.

For `full-geom`, the total number of CE evaluations is

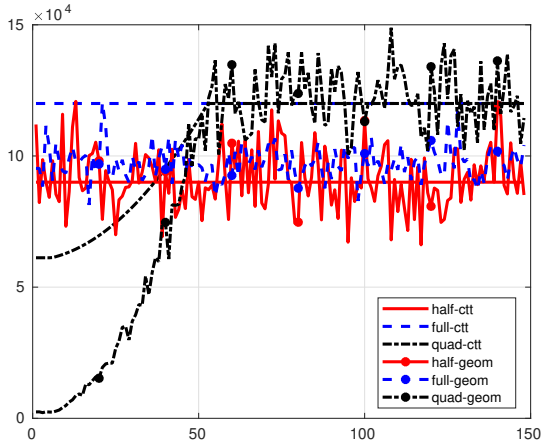
- n for the computation of S_t
- and the $2x$b multiplied by the mean number of ξ_t.

For `full-ctt`,

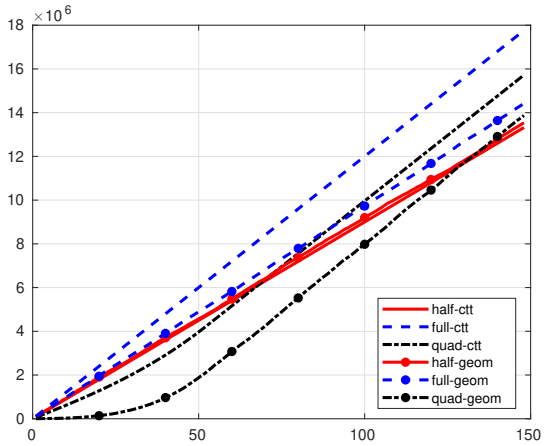
- n for the computation of S_t
- and the $2x\mathsf{b}$ multiplied by k_{in} which is constant over the inner loops.

For `quad-ctt`,

- $\lfloor(\min(n, \max(n/100, 6t^2)))\rfloor$ for the computation of S_t
- and the $2x\mathsf{b}$ multiplied by the mean number of ξ_t.



(a) Per epoch

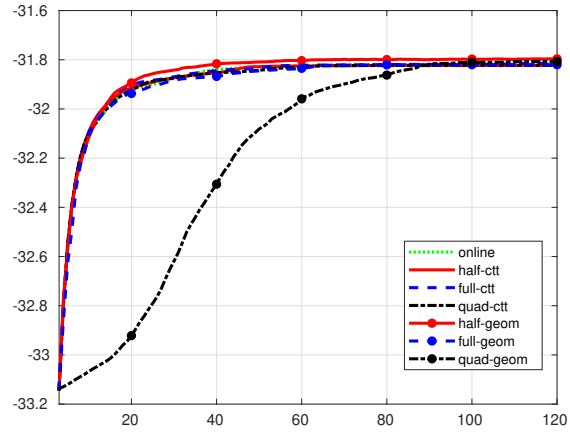


(b) cumulated

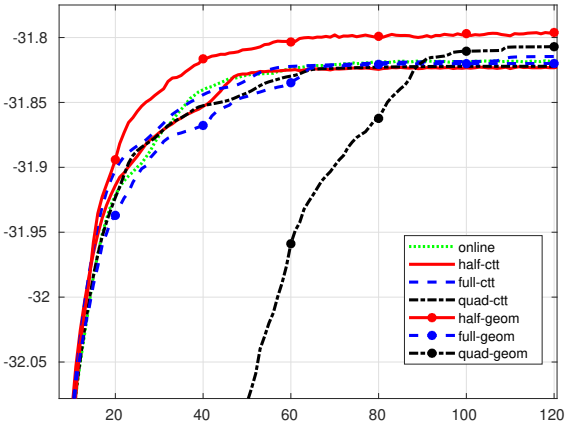
Fig. 1. Number of CE evaluations: per epochs (top) and cumulated (bottom).

Figure 2 displays the mean value of the normalized log-likelihood $n^{-1}\sum_{i=1}^n \mathcal{L}_i \circ \mathsf{T}(\widehat{S}_{t,\Xi_t})$ vs the number of epochs t; this mean value is computed over 30 independent runs of the algorithms. Figure 3 displays the same analysis as in Figure 2 for two different strategies for the number of inner loops ξ_t (on the top, ξ_t is constant and equal to k_{in}; on the bottom, ξ_t is a geometric distribution with expectation k_{in}), and different strategies for the initialization $\mathcal{E}_t$.

Figure 4 displays the quantiles 0.25, 0.5 and 0.75 of the distribution of $\|h(\widehat{S}_{t,\Xi_t})\|^2$; the quantiles are estimated over 30 independent



(a) vs epoch 1 to 120



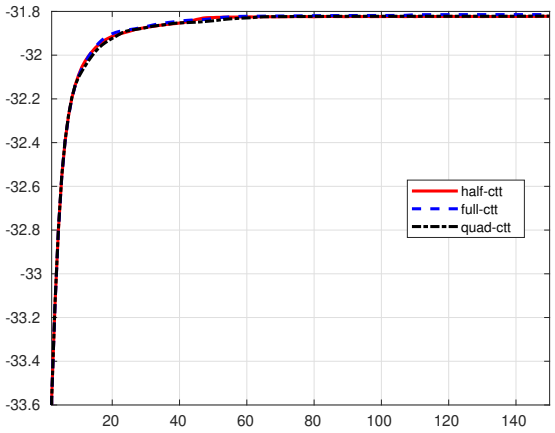
(b) the first epochs are discarded

Fig. 2. The mean value, computed over 30 independent runs of the algorithms, of the normalized log-likelihood $n^{-1}\sum_{i=1}^n \mathcal{L}_i$. The plot shows its value vs the number of epochs.

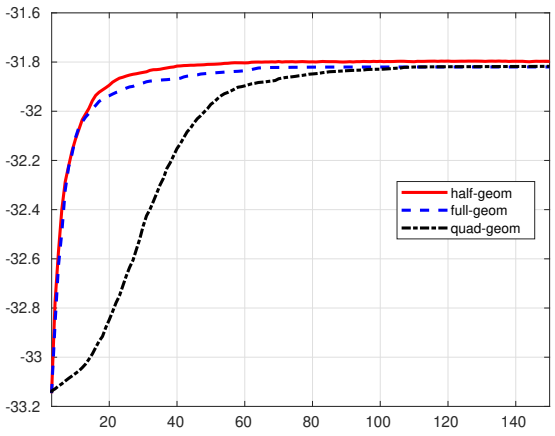
runs of the algorithm. Here the case ξ_t is constant is considered, with different strategies for the initialization $\mathcal{E}_t$.

Figure 5 displays the same analysis as in Figure 4, except that ξ_t is a geometric random variable with expectation k_{in}.

Figure 6 display Figure 4 and Figure 5 on the same plots. The quantiles of the distribution of $\|h(\widehat{S}_t)\|^2$ vs the number of epochs when $\{\widehat{S}_t, t \geq 0\}$ is obtained by `online-EM`.

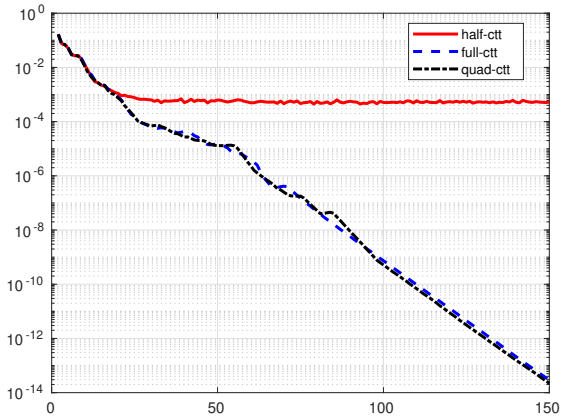


(a) Case of SPIDER-EM (fixed number of inner loop $\xi_t = k_{\text{in}}$)

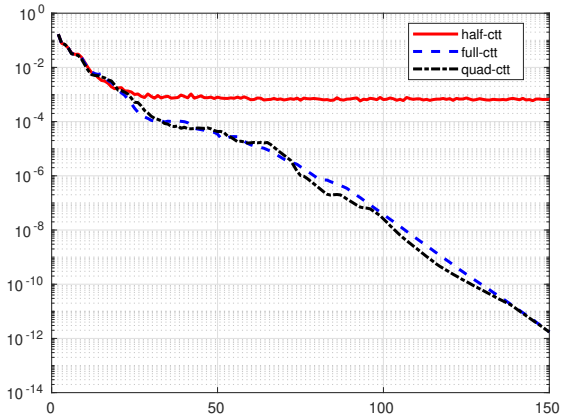


(b) Case of Geom-SPIDER-EM

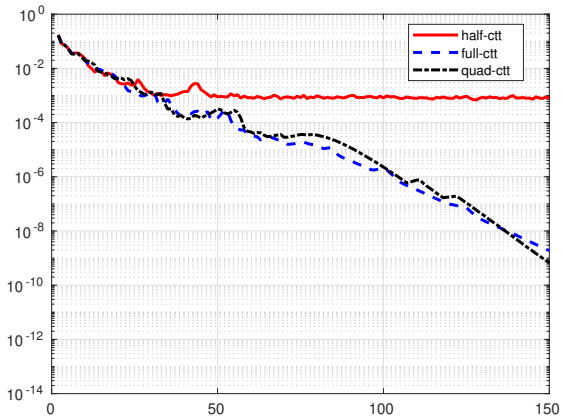
Fig. 3. The mean value, computed over 30 independent runs of the algorithms, of the normalized log-likelihood $n^{-1}\sum_{i=1}^{n}\mathcal{L}_i$. The plot shows the mean value vs the number of epochs, starting from epoch #2



(a) Quantile 0.25

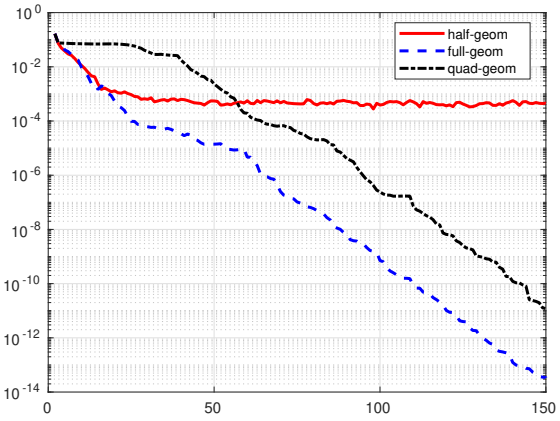


(b) Quantile 0.50

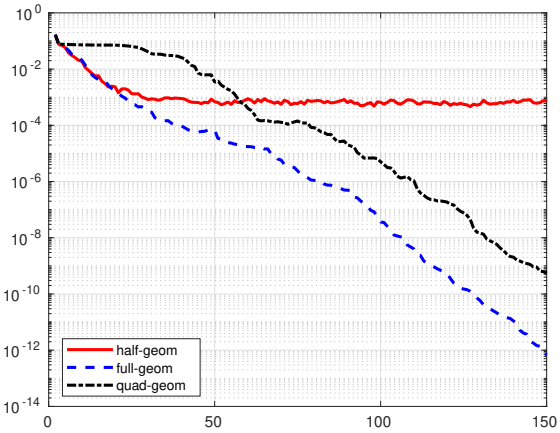


(c) Quantile 0.75

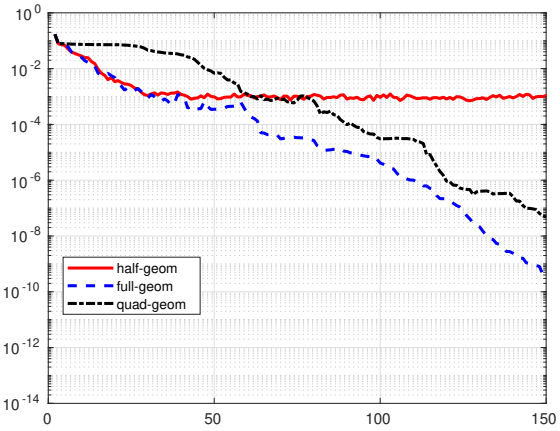
Fig. 4. Case SPIDER-EM. Quantiles of the distribution of $\|h(\widehat{S}_{t,\Xi_t})\|^2$, estimated over 30 independent runs, vs the number of epochs t.



(a) Quantile 0.25

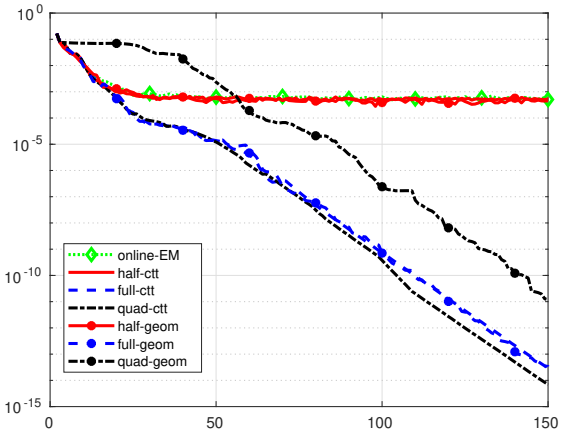


(b) Quantile 0.50

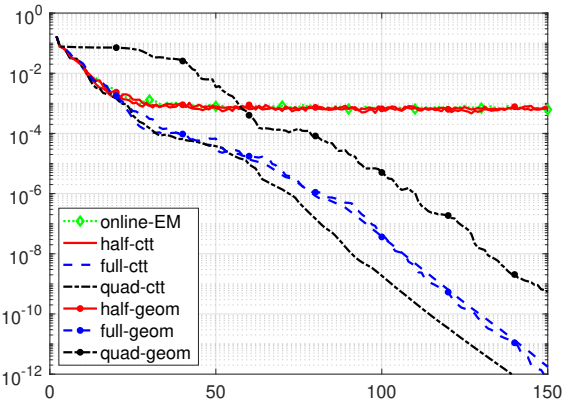


(c) Quantile 0.75

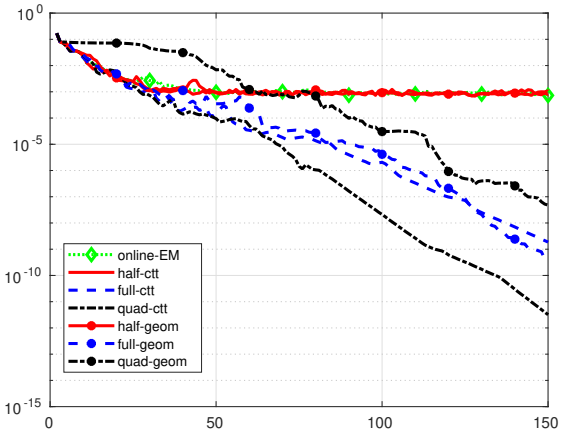
Fig. 5. Case Geom-SPIDER-EM. Quantiles of the distribution of $\|h(\widehat{S}_{t,\Xi_t})\|^2$, estimated over 30 independent runs, vs the number of epochs t



(a) Quantile 0.25



(b) Quantile 0.50



(c) Quantile 0.75

Fig. 6. Comparison of Online-EM, SPIDER-EM and Geom-SPIDER-EM. Quantiles of the distribution of $\|h(\widehat{S}_{t,\Xi_t}\|^2$, estimated over 30 independent runs, vs the number of epochs t.

3. REFERENCES

[1] G. Fort, E. Moulines, and H.T. Wai, "A Stochastic Path-Integrated Differential EstimatoR Expectation Maximization Algorithm," in *Advances in Neural Information Processing Systems 34*. Curran Associates, Inc., 2020.